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On Integer Programming Models for the Multi-Channel PMU Placement Problem and Their Solution

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Abstract Phasor Measurement Units (PMUs) are starting to see increased deployment, enabling accurate measurement of power grid electrical properties to determine system health. Due to the costs associated with PMU acquisition and maintenance, it is practically important to place the minimum number of PMUs in order to achieve system complete observability. In this paper, we consider a variety of optimization models for the PMU placement problem that addresses more realistic assumptions than simple infinite-capacity placement models. Specifically, instead of assuming that a PMU can sense all lines incident to the bus at which it is placed, we impose the more realistic assumption that PMUs have restricted channel capacity, with per-unit cost given as a function of channel capacity. The optimization objective is then to minimize the total cost of placed PMUs, in contrast to their number. Further, we leverage the zero-injection bus properties to reduce the quantity and cost of placed PMUs.

In formulating our optimization models, we identify a close relationship between the PMU placement problem (PPP) and a classic combinatorial problem, the set cover problem (SCP). If channel capacity limits are ignored, there is a close relationship between the PPP and the dominating set problem (DSP), a special case of the SCP. Similarly, when measurement redundancy is imposed as a design requirement, there is a close relationship between the PPP and the set multi-cover problem (SMCP), a generalized version of the SCP. These connections to well-studied combinational problems are not well-known in the power systems literature, and can be leveraged to improve solution algorithms.

We demonstrate that more realistic, high-fidelity PPP optimization models can be solved to optimality using commercial integer programing solvers such as CPLEX.

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Specifically, run-times for all test cases, ranging from IEEE 14-bus to 300-bus test systems, are less than a second. This result indicates that the size of system that can be analyzed using state-of-the-art solvers is considerable. Further, our results call into question the need for problem-specific heuristic solution algorithms for the PPP, many of which have been proposed over the past decade. Finally, we analyze cost versus performance tradeoffs using our PPP optimization models on various IEEE test systems.

Keywords PMU placement · Multi-channel PMUs · Set cover problem · Dominating set problem · Set multi-cover problem · Integer programming

1 Introduction

Phasor networks are seeing increased world-wide deployment, in order to improve stability, state estimation, monitoring and protection, and control and operation in power systems. Phasor Measurement Units (PMUs) are a keystone technology in phasor networks, and their deployment and maintenance costs are a major driver in phasor network design. Consequently, the problem of minimizing the cost of PMU placements in phasor network design is of significant practical interest, and as a result has seen widespread attention from the research community.

PMUs are placed at buses, i.e., substations to which transmission lines, loads, and generators are connected. Due to the high cost of PMUs and the lack of communication facilities at some substations (further increasing the cost of PMU placement at a bus), it is important to minimize the number of PMUs placed while at the same time retaining the ability to monitor the entire power system. A PMU measures the voltage phasor of the bus at which it is placed, and the current phasor of each line incident to such buses. The corresponding buses and lines are said to be *observed*. The *PMU Placement Problem* (PPP) is defined as the problem of minimizing the total number of PMUs placed while ensuring that all buses and lines in a power system are observed. The PPP is conceptually related to several classic combinatorial optimization problems. For example, the dominating set problem requires all vertices in a network to be observed, while the vertex cover problem requires all edges to be observed. However, Ohm's law and Kirchoff's current law impose additional constraints that differentiate the PPP from these classical optimization problems.

The PPP has been widely studied in last decade. The general PPP is known to be NP-hard [1–3]. The computational challenge has motivated the introduction of many heuristic solution methods, including genetic algorithms, tabu search, simulated annealing, and particle swarm optimization [4]. Solution methods based on integer programming (IP) models have also been considered [5–9,4], as the PPP subject to special network topologies [1,2]. Recently, the probabilistic and reliability-based research for PMU placement is considered in the development of wide-area measurement system [10,11]. On the other hand, another new trend in PMU placement is to consider the variable-cost PMUs [12].

These prior studies on the PPP generally assume that an individual PMU can sense all lines incident to the bus at which it is placed. However, in practice PMU manufacturers produce several types of PMUs, with different functions and varying channel properties [13, 14]. Therefore, instead of assuming unlimited channel capacity, we impose the more realistic assumption of restricted channel capacities, with per-unit cost given as a function of channel capacity. Our focus here is then on the general *Multi-Channel PMU Placement Problem* (MC-PPP), in which the objective is to minimize the total cost of PMUs required to achieve complete system observability. To reduce the total cost, we leverage the well-known zero-injection property (based on Kirchoff's current law) in our optimization models [9, 5, 6, 4]. The basic IP model we used in this paper for considering zero-injection property is based on [9].

The MC-PPP has been previously investigated in the literature. For example, [15, 16] considered a special case of the MC-PPP in which PMUs are of unit channel capacity; such PMUs are known as single-channel or branch PMUs. However, this approach cannot be generalized to multi-channel PMU placement. In [17, 18], an integer programming formulation was proposed to solve the MC-PPP, and analyzed on a 14-bus test system. Several approaches to modeling the zero-injection bus property were considered. However, each of these approaches fails to guarantee optimal solutions for power systems with a general network topology. In [19], as an improvement of [18] without any general formulations, it accounted for the number of available channels for the chosen types of PMUs and also it presented upper limit by considering the graph sparsity. In [20], an IP model was proposed based on several complicated matrices, to consider zero-injection buses.

Other researchers have considered limits on the number of PMU channels [21, 22], but these efforts have relied on heuristic procedures to obtain solutions. As a result, no optimality guarantees are provided. [13] similarly considered channel limits, also solved with heuristic procedures, but failed to leverage the zero-injection bus property. Finally, although [20] leverages the zero-injection bus property, the formulation requires significant changes to adapt to different types of PMUs with varying numbers of channels.

The primary contributions of this paper to the PMU placement literature are as follows: (1) we establish a relationship between PMU placement optimization models and several classical combinatorial optimization problems; (2) we propose a general, simple, and easily reproducible integer programming model for multi-channel PMU placement that thoroughly considers the application of the zero-injection bus property; (3) we demonstrate that one can quickly obtain exact, globally optimality solutions to our integer programming model using widely available commercial solvers, calling into question the utility for problem-specific heuristics in this domain; and (4) we show that our integer programming model can be extended to include consideration of PMUs with varying numbers of channel capacities, to be placed into one system in the most cost-effective manner possible.

The reminder of this paper is organized as follows. In Section 2, we first introduce the nomenclature, and Ohm's law and Kirchhoff's current law for observability in power systems. We then present the IP formulations for the MC-PPP with and without consideration of zero-injection buses. In Section 3, we introduce measurement redundancy for buses in the MC-PPP, while in Section 4, a more general MC-PPP is considered with different types of PMUs. In Section 5, numerical experiments are performed and analyzed on six IEEE test systems. Finally, we conclude in Section 6 with a summary of our results.

2 Integer Programming Models for PMU Placement

2.1 Nomenclature

We define a *Power System Graph* (PSG) as an undirected graph $G = (V,E)$. V and E respectively represent sets of buses and transmission lines. Let $V = \{v_1, v_2, \cdots, v_{|V|}\}$, and let an edge $(v_i, v_j) \in E$ denote a transmission line joining buses $v_i, v_j \in V$. Finally, let the cardinality $|V|$ be denoted by n.

Some key structural features of a PSG are given as follows:

- The *adjacency matrix* $A = (a_{ij})_{n\times n}$: Let $a_{ij} = 1$ if $i = j$ or buses i and j are connected by a transmission line $(v_i, v_j) \in E$, and $a_{ij} = 0$ otherwise.
- The *neighborhoods* $N(v_i)$, $N[v_i]$ of a vertex v_i: $N(v_i) = \{v_j \in V | (v_i, v_j) \in E\}$ denotes the open (i.e., excluding v_i) neighborhood of v_i; $N[v_i] = N(v_i) \cup \{v_i\}$ denotes the closed (i.e., including v_i) neighborhood of vertex v_i.
- The *degree* d_i of a vertex v_i: $d_i = |N(v_i)|$ denotes the number of vertices within the open neighborhood of v_i. Let the maximum degree of G be given by $\Delta(G) := \max_{i:v_i\in V} d_i$, and the minimum degree be given by $\delta(G) := \min_{i:v_i\in V} d_i$. If $\delta(G) = 0$, there exists at least one isolated vertex in G. However, because isolated buses are meaningless in the context of power systems analysis, we assume that $\delta(G) \geq 1$.
- The set V_Z of zero-injection buses: The set of zero-injection buses is denoted by $V_Z = \{v_i \in V : Z_i = 1\}$, where $Z_i = 1$ indicates that vertex v_i is a zero-injection bus; $Z_i = 0$ otherwise. Clearly, $V_Z \subseteq V$. A zero-injection bus is a transshipment bus in the system, lacking generation and load. The zero-injection bus property, described subsequently, can be leveraged in order to reduce the number of PMUs required to achieve complete observability of a power system.

The number of lines incident to a bus that a PMU placed at that bus can sense is known as the *channel capacity* of the PMU. In the MC-PPP, the optimization objective is to place PMUs with variable channel capacities on buses of G such that the aggregate set of PMUs can observe the voltage phasors of all buses $v \in V$ and the current phasors of all transmission lines $e \in E$. We now introduce notation related to multi-channel PMUs and optimization models for the MC-PPP:

- An *L-Channel* PMU: Suppose L is a positive non-zero integer. A PMU placed on bus v_i with channel capacity L can measure the voltage phasor of v_i and the current phasors of L lines incident to it. By Ohm's law (as discussed in next section), the voltage phasors of L neighbors of v_i can additionally be inferred. Therefore, if a bus v_i has degree equal to or less than L (i.e., $d_i \leq L$), an L-channel PMU placed at this bus will be sufficient to observe all buses within $N[v_i]$ and all lines incident to v_i. Otherwise, if $d_i > L$, an L-channel PMU placed on v_i can only observe v_i and a subset of $N(v_i)$ with cardinality L, in addition to lines between v_i and this subset. We denote the cost for a PMU with channel capacity L as c_L.

- The number r_i^L of combinations (subsets) of set $N(v_i)$ that an L-channel PMU placed on bus v_i can observe is given by:

$$r_i^L = \begin{cases} 1, & \text{if } d_i \leq L \\ \binom{d_i}{L}, & \text{if } d_i > L. \end{cases}$$

  The total number of these combinations is given by: $m = \sum_{i=1}^{n} r_i^L$, where $\binom{d_i}{L} = d_i!/(L!(d_i - L)!)$.
- The subsets $S_{i1}, S_{i2}, \cdots, S_{ir_i^L}$ of incident lines at bus v_i that a PMU with channel capacity L can observe is given as follows:
  - If $r_i^L = 1$, the subset of incident lines that a PMU with channel capacity L placed at bus v_i that can be observed is $S_{i1} = N[v_i]$.
  - If $r_i^L \geq 2$, the subset of incident lines that a PMU with channel capacity L can be observed is an L-cardinality subset of $N(v_i)$ with the bus v_i.
  - The subsets at v_i are $S_{i1}, S_{i2}, \cdots, S_{ir_i^L}$. Bus v_i is called as the *center* of subset S_{ir} and it is the bus where PMU should be placed to observe S_{ir} $(r = 1, 2, \cdots, r_i^L)$.
- *Family of subsets* at all buses within V:

$$\mathscr{S}_L = \{S_{11}, \cdots, S_{1r_1^L}, S_{21}, \cdots, S_{2r_2^L}, \cdots, S_{n1}, \cdots, S_{nr_n^L}\}$$

  - Let $b_{ijr} = 1$ denote that v_i belongs to the subset S_{jr}, and $b_{ijr} = 0$ otherwise. Obviously, $b_{iir} = 1$ for any $v_i \in V$. Given a PSG G, this family is only related to the channel L. These parameters form a *containment matrix* $B = (b_{ijr})_{n \times n \times r_i}$, to state that whether a bus is in some subset or not.

With an unlimited number of channels, a PMU placed on bus v_i can observe all buses within $N[v_i]$. However, with limited channel capacity L, a PMU at v_i can only observe a subset of $N[v_i]$. Therefore, to completely observe all buses of the system G, we have to choose some subsets within the family $\mathscr{S}_L$ such that PMUs placed on their centers can guarantee complete observability of the power system.

Let $x_s \in \{0, 1\}$ denote a binary decision variable such that $x_s = 1$ if a PMU is placed on the center of subset S_s, and $x_s = 0$ otherwise, where the index s is in the set of indices $\{11, \cdots, 1r_1^L, 21, \cdots, 2r_2^L, \cdots, n1, \cdots, nr_n^L\}$ based on the family $\mathscr{S}_L$. The *Placement vector* x is formed by $x = (x_1, \cdots, x_s, \cdots, x_m)^T \in \{0, 1\}^m$.

Now, we consider an illustrative example using the IEEE 14-Bus system with 14 buses and 20 lines (its topological structure is shown in Fig. 1). Bus 7 is the only zero-injection bus, i.e., $V_Z = \{7\}$.

Assume that the channel capacity for PMUs is 4, i.e., $L = 4$. By the notations and methods introduced above, we have that $r_4^L = 5$, all other r_i^L are 1, and the total number of combinations is $m = \sum_{i=1}^{14} r_i^L = 18$. The family of subsets that a PMU with channel capacity 4 is given as:

$$\begin{aligned}\mathscr{S}_4 = \{&\{1,2,5\},\{2,1,3,4,5\},\{3,2,4\},\{4,2,3,5,7\},\{4,2,3,5,9\},\\ &\{4,2,3,7,9\},\{4,2,5,7,9\},\{4,3,5,7,9\},\{5,1,2,4,6\},\{6,5,11,12,13\},\\ &\{7,4,8,9\},\{8,7\},\{9,4,7,10,14\},\{10,9,11\},\{11,6,10\},\{12,6,13\},\\ &\{13,6,12,14\},\{14,9,13\}\},\end{aligned}$$

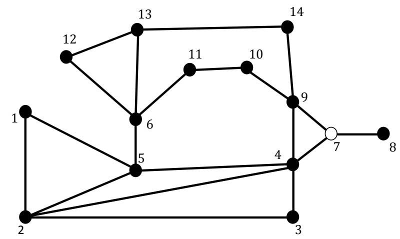

Fig. 1 Topological Structure of the IEEE 14-Bus System

where the center of each subset is assumed to be the first element of the subset.

At bus 4 with $r_4^L = 5$, a PMU can observe exactly all buses within only one set of $\{4,2,3,5,7\},\{4,2,3,5,9\},\{4,2,5,7,9\},\{4,3,5,7,9\}$. By the family $\mathscr{S}_4$, the containment matrix formed by b_{ijr} can be easily obtained.

2.2 Terminology

Two fundamental properties that can be used to reduce the number of PMUs required to achieve complete observability in a power system are as follows:

- **Ohm's law:** The current I_{ij} through a conductor between two points is directly proportional to the voltage $V_i - V_j$ across the two buses v_i, v_j, and is inversely proportional to the resistance R_{ij} between them, i.e., $I_{ij} = (V_i - V_j)/R_{ij}$;
- **Kirchhoff's current law:** At any bus v_i in an circuit, the sum of currents flowing into v_i is equal to the sum of currents flowing out of it, i.e., $\sum_{j:v_j \in N(v_i)} I_{ij} = 0$. Therefore, if all I_{ij} ($v_j \in N(v_i)$) are observed except one (say j'), then $I_{ij'} = -\sum_{j:v_j \in N(v_i), j \neq j'} I_{ij}$, i.e., $I_{ij'}$ is also observed. This law is only applied at zero-injection buses.

In the following, we assume the state variable to be observed on bus v_i is the voltage V_i, and on line (v_i, v_j) is the current I_{ij}. As discussed above, a PMU with channel capacity L placed at bus v_i can observe one subset S_{ir} of $\mathscr{S}_L$. More specifically, such a PMU can directly measure the voltage V_i of bus v_i and the current I_{ij} for $v_j \in S_{ir} \setminus \{v_i\}$. By Ohm's law, the voltage for bus $v_j \in S_{ir} \setminus \{v_i\}$ can be computed by $V_j = V_i - I_{ij}R_{ij}$. Thus, all buses within S_{ir} can be observed by the PMU placed on the center v_i of S_{ir}.

Therefore, a bus v_i of a PSG can be observed in one of the following ways:

(a) A PMU is placed on v_i;
(b) The bus v_i is contained within the subset S_{jr} ($j \neq i$), and a PMU is placed on the center of this subset (by Ohm's law);
(c) The bus v_i is one of the buses within $N[v_j]$ ($(v_j, v_i) \in E$) or $N[v_i]$, while all other buses within the set $N[v_j]$ or $N[v_i]$ are observed (by Ohm's law and Kirchhoff's current law).

Consider the example shown in Fig. 1, which has bus 7 as the only zero-injection bus. If all of the buses within $\{7,4,8,9\}$ except one are observed, this unobserved bus is indirectly observed by Kirchhoff's current law.

By Ohm's law, if the two terminal buses of a line are observed, the line is also observed. Therefore, we have the following:

Lemma 1 *The complete observability of all buses guarantees the complete observability of all lines.*

The objective of the MC-PPP is to find a placement $\mathscr{P} \subseteq \mathscr{S}_L$ with minimum $|\mathscr{P}|$ such that $\bigcup_{S_s \in \mathscr{P}} S_s = V$ in the power system graph $G = (V,E)$. Here a placement for $S_s \in \mathscr{P}$ indicates that a PMU with channel L is placed on the center of S_s.

2.3 An IP Model for the Multi-Channel PMU Placement Problem without Considering the Zero-Injection Buses

Let f_i be the number of times that bus v_i is observed. Without considering Kirchhoff's current law applied on the zero-injection buses, f_i can be expressed as:

$$f_i = \sum_{s:v_i \in S_s} x_s = \sum_{j=1}^{n} \sum_{r=1}^{r_j^L} b_{ijr} x_{jr}, \tag{1}$$

where $\sum_{s:v_i \in S_s} x_s$ denotes that the number of times that bus v_i is in the observed family of $\mathscr{S}_L$, which can be expressed by $\sum_{j=1}^{n} \sum_{r=1}^{r_j^L} b_{ijr} x_{jr}$. For complete observability of all buses, the inequality:

$$f_i \geq 1$$

must hold. The objective of the MC-PPP is to minimize the total cost of PMUs with homogeneous channel capacity L, which can be expressed as $\sum_{j=1}^{n} \sum_{r=1}^{r_j^L} c_L x_{jr}$. An IP formulation for the MC-PPP is then given as follows:

[MC-PPP] :

$$\min \sum_{j=1}^{n} \sum_{r=1}^{r_j^L} c_L x_{jr} \tag{2a}$$

$$s.t. \begin{cases} \sum_{j=1}^{n} \sum_{r=1}^{r_j^L} b_{ijr} x_{jr} \geq 1, \ i = 1, \cdots, n \\ x_{jr} \in \{0,1\}, \ j = 1, \cdots, n, \ r = 1, \cdots, r_j^L \end{cases} \tag{2b}$$

In this formulation, all PMUs with channel capacity L are assumed to have identical costs. The resulting integer program can be solved by commercial solvers, such as CPLEX, to obtain exact solutions directly.

We now discuss several aspects of the MC-PPP formulation:

2.3.1 Relation to the Set Cover Problem

Let V be a set of elements and $\mathscr{S}_L = \{S_{11}, \cdots, S_{1r_1^L}, S_{21}, \cdots, S_{2r_2^L}, \cdots, S_{n1}, \cdots, S_{nr_n^L}\}$ be the family formed by some subsets of V. The MC-PPP is equivalent to the *set cover problem*, i.e., the problem of identifying the smallest number of subsets in $\mathscr{S}_L$ whose union contains all elements in the set V. The decision variable x_s can be considered as whether the subset S_s is selected to be in the union.

The set cover problem is one of Karp's 21 problems shown to be NP-complete in 1972 [23,24], and many algorithms are available for solving this problem [25].

2.3.2 Case $L = 1$: Single-Channel/Branch PMUs

A single-channel or branch PMU is a PMU with $L = 1$, which can only observe two adjacent buses and the single line connecting these two buses.

When $L = 1$ and under the assumption $\delta(G) \geq 1$, there are $r_i^L = d_i$ subsets at bus v_i, and a total of $m = \sum_{i=1}^n r_i^L = \sum_{i=1}^n d_i = 2|E|$ subsets in $\mathscr{S}_L$. Additionally, all subsets in $\mathscr{S}_L$ have the cardinality 2. For the set $\{v_i, v_j\}$ formed by two endpoints of an edge $(v_i, v_j) \in E$, there are two corresponding subsets $S_{ir'}, S_{jr''}$ in $\mathscr{S}_L$.

On the other hand, if a PMU is placed on v_i, bus v_j is observed and vice versa. Given a cost minimization objective, at most one of these vertices will be chosen to place a PMU, i.e., $x_{ir'} + x_{jr''} \leq 1$. To reduce the number of decision variables in formulation (2), one variable for each edge is sufficient to formulate the problem under this assumption. In fact, we have shown that the formulation in [15] is a special case of our formulation.

2.3.3 Case $L \geq \Delta(G)$

The number of combinations at bus v_i is $r_i^L = 1$ for any $v_i \in V$, and the total number is $m = \sum_{i=1}^n r_i^L = n$. In this case, we can establish a relationship between the MC-PPP and the *dominating set problem* (DSP). The DSP is stated as follows: For a graph $G = (V, E)$, a dominating set is a subset D of V such that every vertex not in D is joined to at least one member of D by some edge. The minimum dominating set problem is to find a dominating set D^* with smallest cardinality. The domination number $\gamma(G) = |D^*|$. Assume $x = (x_1, \cdots, x_n)^T$ is a decision vector where $x_i \in \{0,1\}$ indicates whether vertex v_i is in the dominating set ($x_i = 1$) or not ($x_i = 0$). An IP formulation for DSP is as follows:

$$\min \sum_i x_i \tag{3a}$$

$$s.t. \sum_j a_{ij} x_j \geq 1,\ x_i \in \{0,1\},\ i = 1, \cdots, n \tag{3b}$$

The DSP is also a classic combinatorial optimization problem, and is known to be NP-hard [26].

For the MC-PPP in (2), when $L \geq \Delta(G)$, the family of subsets is $\mathscr{S}_L = \{N[v_i] : v_i \in V\}$. Therefore, only exactly one decision variable is required for each subset with center at v_i. In this case, the formulation (2) is equivalent to formulation (3).

Therefore, when $L \geq \Delta(G)$, the optimal value of MC-PPP will remain the same. This corresponds to the PMU placement problem widely studied in the literature, which lacks channel limits and does not consider the zero-injection bus property.

When $L = 4$, solving formulation (2) for the system in Fig. 1, yields PMUs placed on buses $2, 6, 7$ and 9 of subsets $\{2,1,3,4,5\}$, $\{6,5,11,12,13\}, \{7,4,8,9\}$, and $\{9,4,7,10,14\}$. The PMU placed on the corresponding center can observe all buses within that subset. For example, the PMU on bus 6 observes buses 6,5,11,12 and 13. This result does not consider the zero-injection bus 7 and at least 4 PMUs with channel capacity 4 are needed.

2.4 Modeling the Zero-Injection Buses in the MC-PPP

We have shown that the MC-PPP is closely related to the set cover problem and the dominating set problem in special cases. Ohm's law is actually used in these discussions. However, Kirchhoff's current law has not been applied. In the following, we apply Kirchhoff's current law to every zero-injection bus to reduce the required number of PMUs for complete observability of a power system.

Let $y_{ij}, y_{ji} \in \{0,1\}$ be and auxiliary variable for all lines $(v_i, v_j) \in E$. This method was introduced in [9] and [4]. After adding the auxiliary constraints:

$$\sum_{j=1}^{n} a_{ij} y_{ij} = Z_i, \ i = 1, \cdots, n \tag{4a}$$

$$y_{ij} = 0, \ \forall i, j \text{ with } a_{ij} = 0 \text{ or } i \notin V_Z \tag{4b}$$

$$y_{ij} \in \{0,1\}, \ i, j = 1, \cdots, n \tag{4c}$$

two cases can be distinguished: (1) if v_i is not a zero-injection bus ($Z_i = 0$), all $y_{ij} = 0$ for lines incident to i; (2) if v_i is a zero-injection bus ($Z_i = 1$), all $y_{ij} = 0$ except for one $y_{ij'}$; this non-zero term can be added to $f_{j'}$, which increases the observability of bus $v_{j'}$. Therefore, by considering the zero-injection buses connected to v_i, f_i in formulation (1) becomes

$$f_i = \sum_{j=1}^{n} \sum_{r=1}^{r_j^L} b_{ijr} x_{jr} + \sum_{j=1}^{n} a_{ij} Z_j y_{ji}, \ i = 1, \cdots, n. \tag{5}$$

There are other methods for modeling zero-injection buses. For example, nonlinear terms were added in [5], and auxiliary variables were added for buses in [6]. In this formulation, every zero-injection bus provides an observation of some bus by (4a), and further reductions are impossible.

The IP formulation for the MC-PPP with consideration of zero-injection buses is then given as follows:

[MC-PPP]^0 :

$$\min \sum_{j=1}^{n} \sum_{r=1}^{r_j^L} c_L x_{jr} \tag{6a}$$

$$s.t. \begin{cases} \sum_{j=1}^{n} \sum_{r=1}^{r_j^L} b_{ijr} x_{jr} + \sum_{j=1}^{n} a_{ij} Z_j y_{ji} \geq 1, \ i = 1, \cdots, n \\ \sum_j a_{ij} y_{ij} = Z_i, \ i = 1, \cdots, n \\ y_{ij} = 0, \ \forall i, j \text{ with } a_{ij} = 0 \text{ or } i \notin V_Z \\ x_{jr}, y_{ij} \in \{0,1\}, \ i, j = 1, \cdots, n, \ r = 1, \cdots, r_j^L \end{cases} \tag{6b}$$

Similarly, when $L \geq \Delta(G)$, this formulation is for the general PMU placement problem without considering channel limits.

When $L = 4$, solving the formulation in (6) for the system shown in Fig. 1, yields PMUs placed on buses 2,6 and 9 of subsets $\{2,1,3,4,5\}, \{6,5,11,12,13\}$, and $\{9,4,7,10,14\}$. Since buses 7,4 and 9 within $\{7,4,8,9\}$ are all observed by the PMU on bus 9, only bus 8 is not observed. By applying Kirchhoff's current law to the zero-injection bus 7, bus 8 is observed. Comparing with the result by (2), we can reduce the number of required PMUs by one for this system.

3 PMU Placement with Measurement Redundancy

Measurement redundancy is defined as the observability counts for a bus minus one. When monitoring a power system, each bus has to be observed at least once for complete observability. However, for cross-validation analysis [3] and some contingencies, it is desirable for every bus to have some kind of measurement redundancy. In some papers [6,3], instead of minimizing the number of PMUs, observability is maximized for given number of PMUs.

The *redundancy for bus* v_i is defined as $f_i - 1$. Assume R_b is the redundancy requirement for all buses. We introduce the following IP formulation for the MC-PPP considering measurement redundancy for buses:

[MC-PPP]^R :

$$\min \sum_{j=1}^{n} \sum_{r=1}^{r_j^L} c_L x_{jr} \tag{7a}$$

$$s.t. \begin{cases} \sum_{j=1}^{n} \sum_{r=1}^{r_j^L} b_{ijr} x_{jr} + \sum_{j=1}^{n} a_{ij} Z_j y_{ji} - 1 \geq R_b, \ i = 1, \cdots, n \\ \sum_j a_{ij} y_{ij} = Z_i, \ i = 1, \cdots, n \\ y_{ij} = 0, \ \forall i, j \text{ with } a_{ij} = 0 \text{ or } i \notin V_Z \\ x_{jr}, y_{ij} \in \{0,1\}, \ i, j = 1, \cdots, n, \ r = 1, \cdots, r_j^L \end{cases} \tag{7b}$$

For bus v_i, the summation $\kappa_i := \sum_{j=1}^{n}\sum_{r=1}^{r_j^L} b_{ijr}$ denotes the number of subsets in $\mathscr{S}_L$ that contain v_i. Let $\kappa(G,L) = \min_{i=1,\cdots,n}\kappa_i$ denote the minimum number of subsets that buses in V can belong to those subsets in $\mathscr{S}_L$. To satisfy R_b redundancy for bus v_i, considering the relations between the redundancy R_b and $\kappa(G,L)$, there are two cases needed to be addressed:

- If $\kappa(G,L) \geq R_b + 1$, $\sum_{j=1}^{n}\sum_{r=1}^{r_j^L} b_{ijr}x_{jr} + \sum_{j=1}^{n} a_{ij}Z_j y_{ji} \geq R_b + 1$ could have a feasible solution and satisfies the redundancy requirement R_b for all buses;
- if $\kappa(G,L) < R_b + 1$, for buses in $\{v_i \in V : \kappa_i \geq R_b + 1\}$, the constraint

$$\sum_{j=1}^{n}\sum_{r=1}^{r_j^L} b_{ijr}x_{jr} + \sum_{j=1}^{n} a_{ij}Z_j y_{ji} \geq R_b + 1$$

  is still valid; for buses in $\{v_i \in V : \kappa_i < R_b + 1\}$, additional PMUs should be placed.

We assume that $R_b + 1 \leq \kappa(G,L)$, and among the test systems, $R_b = 0$ or 1. For $R_b = 0$, this model reduces to the model [MC-PPP]^0.

When $R_b = 1$, PMUs can still have complete system observability given the loss of any one PMU. In case of the loss of any one PMU, buses within the set that this PMU was placed will have one less mechanism for observability, but still have observability larger than or equal to 1. The number of observability mechanisms for all other buses are all still larger than or equal to 2. Thus, the whole system still has complete observability.

For the system shown in Fig. 1, when $R_b = 1$, 8 PMUs with channel capacity 4 are placed on buses $2,4,5,6,7,9,10$ and 13 in order to achieve complete system observability with one redundancy at minimal cost.

Without considering the zero-injection bus property, the constraints in formulation (7) become:

$$\begin{cases} \sum_{j=1}^{n}\sum_{r=1}^{r_j^L} b_{ijr}x_{jr} \geq R_b + 1,\ i = 1,\cdots,n \\ x_{jr} \in \{0,1\},\ j = 1,\cdots,n,\ r = 1,\cdots,r_j^L \end{cases}$$

which, with the objective $\min\sum_{j=1}^{n}\sum_{r=1}^{r_j^L} c_L x_{jr}$, is the *Set Multi-Cover Problem*. This problem is a generalized version of the set cover problem, and requires that the selected subsets can cover every element at least $R_b + 1$ times. For more details of this problem, we refer to [27].

4 Extended Formulation for PMUs with Different Channel Capacities

In Sections II and III, the PMUs for a given system are assumed to have the same channel capacity L. If there are different types of PMUs with different channel capacities that can be installed into a system, this will change the total installation cost.

We assume that PMUs can have channel capacities from $l = 1, 2, \cdots, L$ with different costs, where L is largest number of available channels.

For subsets at bus v_i, we have to consider all combinations/subsets with cardinality $l = 1, 2, \cdots, L$. If $d_i \le L$, there are

$$r_i = 2^{d_i} - 1$$

subsets at bus v_i; If $d_i > L$, there are

$$r_i = \binom{d_i}{1} + \binom{d_i}{2} + \cdots + \binom{d_i}{L}$$

subsets. The family $\mathscr{S} = \cup_{l=1,\cdots,L} \mathscr{S}_l$ now has more number of subsets than $\mathscr{S}_L$. When $L \ge \Delta(G)$, the total number of subsets is given as:

$$\sum_{i=1}^{n} 2^{d_i} - n.$$

Based on this family, we can straightforwardly form the containment matrix $B = (b_{ijr})_{n \times n \times r_i}$.

Similarly, for a subset S_{ir} with one center v_i and $|S_{ir}| - 1$ buses connected to it, a PMU with channel capacity $|S_{ir}| - 1$ placed on v_i will be sufficient to observe all buses within S_{ir}. Assume that the cost to place a PMU on the center of S_{ir} is $c_{ir}(S_{ir})$, and that this cost should reflect the channel capacity $|S_{ir}| - 1$. For example, we can simply assume $c_{ir}(S_{ir}) = |S_{ir}|$, which is equal to the channel capacity of the PMU placed on the center of S_{ir}. The integer programming formulation for the MC-PPP with different types of PMUs and consideration of zero-injection buses can be given as follows:

$$[\textbf{MC}^t\textbf{-PPP}]^0 :$$

$$\min \sum_{j=1}^{n} \sum_{r=1}^{r_j} c_{jr}(S_{jr}) x_{jr} \tag{8a}$$

$$s.t. \begin{cases} \sum_{j=1}^{n} \sum_{r=1}^{r_j} b_{ijr} x_{jr} + \sum_{j=1}^{n} a_{ij} Z_j y_{ji} \ge 1, \; i = 1, \cdots, n \\ \sum_j a_{ij} y_{ij} = Z_i, \; i = 1, \cdots, n \\ y_{ij} = 0, \; \forall i, j \text{ with } a_{ij} = 0 \text{ or } i \notin V_Z \\ x_{jr}, y_{ij} \in \{0, 1\}, \; i, j = 1, \cdots, n, \; r = 1, \cdots, r_j \end{cases} \tag{8b}$$

Given optimization objective (8a), at most one subset at bus v_i will be chosen to place a PMU to satisfy the complete observability requirement. In contrast to (6), this formulation can have as many as $\sum_{i=1}^{n} 2^{d_i} - n$ decision variables in x while (6) has $\sum_{i:d_i \le L} 1 + \sum_{i:d_i > L} \binom{d_i}{L}$ variables in x.

As indicated in Section 1, the PMU placement problem with varying numbers of channel capacities was proposed in [13]. However, heuristic methods were used for obtaining approximate solutions without considering the effects of zero-injection buses. Although [20] considered the zero-injection buses, its original formulation and the associated problem parameters had to be significantly changed to consider PMUs with varying numbers of channels. In our model, if PMUs are only available with channel capacities from the set $\{l_1, l_2, \cdots, l_L\}$ (in the increasing order, with i_k a positive integer for any $k = 1, \cdots, L$), the family of subsets can be generated similarly. Depending on the relation between the degree of a bus and the values within the channel capacity set, we generate the subsets with the corresponding cardinalities. Precisely, if $d_i > l_L$, all subsets with cardinality l_k ($1 \leq k \leq L$) should be generated; when $i_k \leq d_i \leq i_{k+1}$ for some k, all subsets with cardinality $i_1, \cdots, i_k$ should be generated. The corresponding family of subsets $\mathscr{S}$ can be generated by these subsets, and similarly the matrix B can be formed. In the numerical experiments of next section, we will compare our results with those reported in [20].

5 Numerical Experiments

The IP formulations for our proposed optimization models are implemented in C++ and solved using CPLEX 12.1 via IBM's Concert Technology 2.9 callable library. All computations were performed on a Linux workstation with a quad-core Intel®Xeon™ 3.60GHz CPU and 8GB of RAM. Each of the experiments reported below (i.e., the solution of a particular instance of a given model) were completed within 1 second of wall clock time. Each model is tested on a variety of standard IEEE test instances.

In Table 2, we report the number of combinations / subsets m for different values of L, for each test instance we consider. The number of combinations generally runs in the hundreds (e.g., for the IEEE 30-bus, 57-bus, 118-Bus, and RTS-96 systems), although the counts reach into the thousands for the larger IEEE 300-bus system. Yet, despite relatively large values of m, CPLEX is able to quickly solve all instances – independent of the optimization model.

In Section 2, we illustrated our optimization models using the IEEE 14-bus test system given PMUs with channel capacity $L = 4$. Here, we expand this analysis by considering six standard IEEE test systems, varying the channel capacities from 1 to $\Delta(G)$ for the corresponding instance. Parallel lines were removed from the original instances, obtained from [28]. In the left half of Table 1, we report key statistics for the various test systems. For each combination of test instance and L, we generated the family $\mathscr{S}_L$, and then formed the containment matrix based on this family. To generate all L-cardinality subsets of $N(v_i)$, we used a simple recursive method. Alternatively, for improved speed, commercial software such as Matlab can be used to generate these subsets directly.

We first consider the minimal number of PMUs obtained on the various test instances, for the **[MC-PPP]**, **[MC-PPP]**^{0}, and **[MC-PPP]**^{R} optimization models. Clearly, the cost in these cases is simply c_L times the corresponding PMU count, and the counts for cases when $L \geq \Delta(G)$ are identical to the results obtained for $L = \Delta(G)$. Table 1 reports results obtained when ignoring the zero-injection bus property (optimization model **[MC-PPP]**, while the left half of Table 3 reports results obtained by considering the application of Kirchhoff's current law to zero-injection buses. Comparing results for a given test instance under the same channel capacity L, we observe – as expected - a significant drop in the number of PMUs required to achieve system

Table 1 Results for Multi-Channel PMU Placement by **[MC-PPP]**

					Optimal number of PMUs when Channel $L=$										
Instance	$\lvert V\rvert$	$\lvert E\rvert$	$\sum_i Z_i$	$\Delta(G)$	1	2	3	4	5	6	7	8	9	10	11
IEEE 14-Bus	14	20	1	5	7	5	4	4	4						
IEEE 30-Bus	30	41	6	7	15	11	10	10	10	10	10				
IEEE 57-Bus	57	78	15	6	29	19	17	17	17	17					
RTS-96	73	108	22	5	37	25	20	20	20						
IEEE 118-Bus	118	179	10	9	61	41	33	32	32	32	32	32	32		
IEEE 300-Bus	300	409	65	11	167	105	91	89	88	88	88	87	87	87	87

Note: The column labeled "$\sum_i Z_i$" reports the total number of zero-injection buses.

observability. Empirically, the reduction in PMUs required is generally less than or equal to the number $\sum_i Z_i$ of zero-injection buses in the system.

In [17] (Table 4.1 and Table 4.2, page 32–33), similar results are reported for all systems with the exception of RTS-96, and [18] (Table I) and [20] (Table I and Table III) report similar results for all systems with the exception of RTS-96 and the IEEE 300-bus. In contrast to the results reported in [17, 18], which were obtained using heuristic solution procedures for consideration of zero-injection bus property, our approach guarantees globally optimal solutions for all test instances. With the exceptions reported below, our results are always less than or equal to the values reported in [17, 18, 20]. This implies that our methods (exact solution via integer programming) generally obtain better solutions. Additionally, our methods still obtain exact solutions for IEEE 300-Bus system in less than 1 second, while the results for this system was not tested in [17, 18, 20]. For experiments with the IEEE 118-Bus system when $L=3$ or 4 (in first half of Table 3), we have results 31 and 29, while the corresponding values in [17, 18] are 30 and 28 (best among all approaches used). Because the actual PMU placements were not reported in [17, 18], we cannot verify the correctness. However, we observe that our results are obtained using a commercial solver, which consistently (with the exception of these cases) obtains superior results.

Next, we consider the results (shown in the second half of Table 3) obtained under the **[MC-PPP]**^{R} model, requiring one redundant observation per bus – which in turn guarantees one redundant observation per line. Comparing the cases without redundancy and with single-observation redundancy, we observe that very large numbers (in most cases, at least double) of additional PMUs are required. This result provides a rigorous quantification of the cost of imposing redundancy requirements for system observability.

We now relax the requirement of homogeneous channel capacities L, and focus on the total cost of PMUs placed under this relaxation. Assume the cost for a PMU with channel capacity L is $c_L = L+1$. Further, assume that a PMU with channel capacity $|S_{ir}|-1$ – required to observe the subset S_{ir} – has cost $c(S_{ir}) = |S_{ir}|$. Results for the minimal cost of PMU placement for the optimization models **[MC-PPP]**^{0} and **[MC^t-PPP]**^{0} are shown in Table 4. For all test systems, we observe that the total cost of PMUs becomes stable when there are more available types of PMUs. Further, when contrasting the minimal costs given a single type of PMU (as reported in the first half of Table 4) versus multiple types of PMUs (asa reported in the second half of Table 4), we observe that the cost of multiple PMUs is always less than the single

Table 2 Number of Combinations/Subsets m for Different Values of L

	L										
Instance	1	2	3	4	5	6	7	8	9	10	11
IEEE 14-Bus	40	47	35	18	14						
IEEE 30-Bus	82	103	104	82	55	36	30				
IEEE 57-Bus	156	171	143	97	67	57					
RTS-96	216	252	178	97	73						
IEEE 118-Bus	358	516	582	508	368	240	160	126	118		
IEEE 300-Bus	818	1131	1193	1202	1126	928	678	472	354	310	300

Table 3 Results for Multi-Channel PMU Placement by **[MC-PPP]**^{0} and **[MC-PPP]**^{R}

L	1	2	3	4	5	6	7	8	9	10	11
Instance	**[MC-PPP]**^{0} Optimal number of PMUs										
IEEE 14-Bus	7	5	4	3	3						
IEEE 30-Bus	13	8	7	7	7	7	7				
IEEE 57-Bus	21	14	12	11	11	11					
RTS-96	26	17	14	14	14						
IEEE 118-Bus	56	37	31	29	28	28	28	28	28		
IEEE 300-Bus	127	85	72	69	69	69	69	68	68	68	68
Instance	**[MC-PPP]**^{R} Optimal number of PMUs when $R_b = 1$										
IEEE 14-Bus	14	9	8	8	8						
IEEE 30-Bus	28	19	16	16	16	16	16				
IEEE 57-Bus	50	33	28	27	27	27					
RTS-96	62	42	33	32	32						
IEEE 118-Bus	115	78	64	63	63	63	64	64	64		
IEEE 300-Bus	290	191	165	162	161	162	164	164	163	163	170

type of PMUs. This observation has the potential for significant impact on practice, in terms of achieving significant cost savings, and more accurately reflects the range of hardware options available for deployment.

Table 4 Minimum Cost of Multi-Channel PMU Placement by **[MC-PPP]**^{0} and **[MC^t-PPP]**^{0}

L	1	2	3	4	5	6	7	8	9	10	11
Instance	**[MC-PPP]**^{0} Objective										
IEEE 14-Bus	14	15	16	15	18						
IEEE 30-Bus	26	24	28	35	42	49	56				
IEEE 57-Bus	42	42	48	55	66	77					
RTS-96	52	51	56	70	84						
IEEE 118-Bus	112	111	124	145	168	196	224	252	280		
IEEE 300-Bus	254	255	288	345	414	483	552	612	680	748	816
Instance	**[MC^t-PPP]**^{0} Objective when max L										
IEEE 14-Bus	14	13	13	13	13						
IEEE 30-Bus	26	24	24	24	24	24	24				
IEEE 57-Bus	42	42	42	42	42	42					
RTS-96	52	51	51	51	51						
IEEE 118-Bus	112	108	108	108	108	108	108	108	108		
IEEE 300-Bus	254	240	237	236	236	236	236	235	235	235	235

Note: Assume that $c_L = L + 1, c(S_{ir}) = |S_{ir}|$.

Finally, In Table 5 and Table 6, we repeat the experiments reported by [20] (Table II and Table IV in [20]), using two types of PMUs with varying channel capacities to obtain complete system observability. Additionally, we also test our models on IEEE 300-Bus system. Table 5 shows the optimal numbers of two types of PMUs (under different cost ratios) without considering the zero-injection bus property, while Table 6 reports results obtained when considering the zero-injection bus property. Comparing our results with those reported in [20], with the exceptions reported below, our results are always require equal or fewer numbers of PMUs. For example, consider the results reported in Table 6 for the IEEE 118-bus test system, when $c_1/c_2 = 5/6$. Under our optimization model, we require 3 PMUs with channel limit 1 and 34 PMUs with channel limit 2 (the associated cost is $43.8c_1$), while results reported in Table IV of [20] indicate that 7 PMUs with channel limit 1 and 31 PMUs with channel limit 2 are required (associated cost is $44.2c_1$). For experiments with the IEEE 57-Bus system when $c_1/c_2 = 4/5$ (in Table 5), we have results 16 and 5 (the associated cost is $22.25c_1$), while the corresponding values in [20] are 7 and 10 (the associated cost is $19.5c_1$). Because the actual PMU placements were not reported in [20], we cannot verify the correctness. However, we observe that our results are obtained using a commercial solver, which consistently (with the exception of this case, and the cases for IEEE 118-Bus when $c_1/c_4 = 4/5$ in Table 5 and also in Table 6) obtains superior results.

Table 5 Optimal Number of Various Types of PMUs for Multi-Channel PMU Placement (w/o consideration of zero-injection buses)

Types of PMUs with Channel Limit	Associated Cost Ratio	IEEE 14-Bus	IEEE 30-Bus	IEEE 57-Bus	RTS-96	IEEE 118-Bus	IEEE 300-Bus
$l_1 = 1$	$c_1/c_2 = 5/6$	1	3	0	2	5	10
$l_2 = 2$		4	8	19	23	36	95
$l_1 = 1$	$c_1/c_3 = 2/3$	1	5	9	5	13	37
$l_2 = 3$		3	5	10	16	23	59
$l_1 = 1$	$c_1/c_4 = 4/5$	1	6	16	12	16	62
$l_2 = 4$		3	4	5	11	19	40
$l_1 = 1$	$c_1/c_4 = 1/3$	7	15	29	37	58	130
$l_2 = 4$		0	0	0	0	1	10
$l_1 = 2$	$c_2/c_5 = 2/3$	3	9	17	21	27	78
$l_2 = 5$		1	1	1	2	7	14
$l_1 = 2$	$c_2/c_5 = 1/2$	5	11	19	25	36	98
$l_2 = 5$		0	0	0	0	2	3

Table 6 Optimal Number of Various Types of PMUs for Multi-Channel PMU Placement (with consideration of zero-injection buses)

Types of PMUs with Channel Limit	Associated Cost Ratio	IEEE 14-Bus	IEEE 30-Bus	IEEE 57-Bus	RTS-96	IEEE 118-Bus	IEEE 300-Bus
$l_1 = 1$	$c_1/c_2 = 5/6$	2	0	0	0	3	13
$l_2 = 2$		3	8	14	17	34	71
$l_1 = 1$	$c_1/c_3 = 2/3$	1	4	5	4	9	27
$l_2 = 3$		3	4	8	11	23	47
$l_1 = 1$	$c_1/c_4 = 4/5$	0	3	9	5	14	45
$l_2 = 4$		3	4	5	9	17	32
$l_1 = 1$	$c_1/c_4 = 1/3$	7	13	21	26	52	105
$l_2 = 4$		0	0	0	0	1	6
$l_1 = 2$	$c_2/c_5 = 2/3$	1	6	12	12	25	61
$l_2 = 5$		2	1	1	3	6	12
$l_1 = 2$	$c_2/c_5 = 1/2$	5	8	14	17	33	74
$l_2 = 5$		0	0	0	0	2	5

6 Conclusions

In this paper, we have studied a variety of general integer programming models for the multi-channel PMU placement problem, focusing on incorporation of more realistic modeling assumptions than are typically considered in the literature. These include consideration of PMUs with limited channel capacities, the availability of PMUs with varying channel capacities, and PMU placement subject to measurement redundancy. We demonstrated that these integer programming models are easily solvable by commercial software systems, to global optimality, and in minimal run-times (less than one second for all tested systems and cases). Our results suggest that heuristic solution methods – which cannot guarantee optimality – are not necessary to solve realistic PMU placement models, and may not be necessary for even larger, real-world systems. Additionally, we establish relationships between the PMU placement problem and some classic combinatorial problems, such as the set cover problem, the dominating set problem, and the set multi-cover problem. Establishing these connections, which are not widely known in the power systems literature, can potentially impact solution methods for PMU placement problems in the future.

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